AN APPROXIMATE METHOD FOR CALCULATING THE BOUNDARY LAYER OF TWO-TEMPERATURE PLASMA AT ELECTRODES WITH LARGE VALUES OF THE HALL PARAMETER

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The processes occurring in the plasma boundary layer at electrodes are described by a complicated system of differential equations with boundary conditions specified on two boundaries. Even when a computer is used, solution of a boundary-value problem of this type presents serious difficulties. Besides, a method is desirable whereby fairly fast estimates can be obtained and the influence of different gross factors analyzed. From this point of view, the development of approximate methods is of great value for theory of the plasma boundary layer.

The method of integral relations [1] is used in the present paper for solving the problem on the boundary layer of a fully ionized two-temperature plasma at electrodes of a channel with crossed E and B fields. In its essential approach, the method follows the same lines as the author's earlier paper [2].

The Problem. To solve the system of equations, of continuity,

$$\frac{\partial}{\partial x}(nu) + \frac{\partial}{\partial y}(nv) = 0 \tag{1}$$

of motion,

$$mn\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\eta_i\frac{\partial u}{\partial y}\right) + j_y B \tag{2}$$

of the ion energy,

$$mnc_p\left(u\frac{\partial T_i}{\partial x} + v\frac{\partial T_i}{\partial y}\right) = \frac{\partial}{\partial y}\left(\lambda_i\frac{\partial T_i}{\partial y}\right) + u\frac{\partial p_i}{\partial x} + v\frac{\partial p_i}{\partial y} + \eta_i\left(\frac{\partial u}{\partial y}\right)^2 + \frac{3km_e}{m}\frac{n}{\tau_e}(T_e - T_i) \tag{3}$$

and of the electron energy,

$$\begin{gathered} mnc_p\left(u\frac{\partial T_e}{\partial x} + v\frac{\partial T_e}{\partial y}\right) = \frac{\partial}{\partial y}\left(\lambda_e\frac{\partial T_e}{\partial y} + \frac{5}{2}\frac{j_y^q}{e}kT_e\right) + u\frac{\partial p_e}{\partial x} + v\frac{\partial p_e}{\partial y} \\ + j_x E_x + j_y(E_y - uB) + \eta_e\left(\frac{\partial u}{\partial y}\right)^2 - \frac{3km_e}{m}\frac{n}{\tau_e}(T_e - T_i) \end{gathered} \tag{4}$$

under the boundary conditions

$$u(x, 0) = v(x, 0) = 0,\ T_i(x, 0) = T_{iw},\ T_e(x, 0) = T_{ew}$$

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$$u(x, \infty) = U_s(x), \; T_i(x, \infty) = T_{is}(x), \; T_e(x, \infty) = T_{es}(x)$$
$$j_y(x, \infty) = j_{ys}(x), \; E_x(x, \infty) = E_{xs}(x), \; j_x(x, \infty) = 0$$

It was shown in [3] that the functions p, j_y, and E_x are constant over the boundary-layer cross-section.*

The current densities are given by

$$j_x = \sigma\left\{E_x - \frac{A_2}{A_1}\left(E_y - uB + \frac{1}{en}\frac{\partial p_e}{\partial y}\right) - \frac{k}{e}\frac{A_4}{A_1}\frac{\partial T_e}{\partial y}\right\} \tag{5}$$

$$j_y = \sigma\left\{E_y - uB + \frac{1}{en}\frac{\partial p_e}{\partial y} + \frac{A_2}{A_1}E_x + \frac{k}{e}\frac{A_3}{A_1}\frac{\partial T_e}{\partial y}\right\} \tag{6}$$

$$j_y^q = \sigma\left\{\frac{A_7}{A_1}\left(E_y - uB + \frac{1}{en}\frac{\partial p_e}{\partial y}\right) + \frac{A_8}{A_1}E_x\right\}. \tag{7}$$

The transport properties η_i, η_e, λ_i, λ_e, σ of the plasma across the magnetic field, and the coefficients A_k, are known functions of T_i, T_e, n, and the Hall parameters H_i and H_e [3].

The profile of the plasma mean velocity per unit mass may be written as the polynomial

$$u^\circ = a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4$$
$$\left(\xi = \frac{1}{\delta}\int_0^y n^\circ dy, \; \delta = \int_0^{\delta_y} n^\circ dy, \; u^\circ = \frac{u}{U_s}, \; n^\circ = \frac{n}{n_s}\right). \tag{8}$$

Here $\delta_y = \delta_y(x)$ is the thickness of the dynamic boundary layer.

The condition on the inner boundary of the layer is obtained from the equation of motion (2), which, when y = 0, gives

$$-\frac{\partial p}{\partial x} + \left[\frac{\partial}{\partial y}\left(\eta_i\frac{\partial u}{\partial y}\right)\right]_w + j_yB = 0. \tag{9}$$

On applying the equation of motion close to the outer boundary of the layer, we get

$$mn_sU_s\frac{dU_s}{dx} = -\frac{dp}{dx} + j_yB. \tag{10}$$

Consequently,

$$\left[\frac{\partial}{\partial y}\left(\eta_i\frac{\partial u}{\partial y}\right)\right]_w = -mn_sU_sU_s'. \tag{11}$$

Using the transformation of (8) from the variable y to ξ, and assuming for simplicity that the product $\eta_i\rho$ is constant across the boundary layer

$$\frac{\eta_i\rho}{\eta_{is}\rho_s} = K_i \qquad (\rho = mn)$$

we obtain

$$\frac{\partial^2 u^\circ}{\partial\xi^2} = -\lambda \qquad \left(\lambda = \frac{\delta^2 mn_s^2U_s'}{n_w\eta_{is}K_i}\right). \tag{12}$$

*The term $-j_yuB$ should appear on the right side of Eq. (23) of [3]. There is a misprint in expressions (8) of [3] for π_{xx}^e, π_{xy}^e, π_{yy}^e. The numerical factor of the third terms on the right side is equal to $10/[3(2+\sqrt{2})]$.

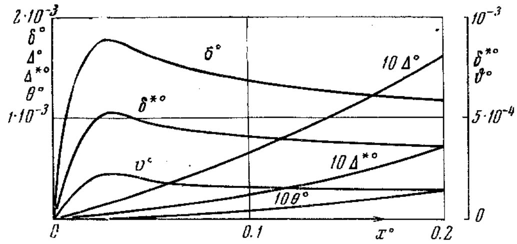

Fig. 1

Assume that

$$u^\circ = 1, \qquad \frac{\partial u^\circ}{\partial \xi} = 0,\ \frac{\partial^2 u^\circ}{\partial \xi^2} = 0 \tag{13}$$

on the outer boundary $\xi = 1$ of the layer.

From (12) and (13),

$$a_1 = \frac{12+\lambda}{6}, \qquad a_2 = 6 - 3a_1, \qquad a_3 = -8 + 3a_1, \qquad a_4 = 3 - a_1 .$$

Let the ion braking temperature and braking temperature gradient be defined by

$$T_i^* = T_i + \frac{u^2}{2c_p}, \quad T_{is}^* = T_{is} + \frac{u_\Delta^2}{2c_p} \quad t_i^* = T_i^* - T_{iw}, \quad t_{is}^* = T_{is}^* - T_{iw} .$$

The profile of the braking temperature gradients will be written as the polynomial

$$t_i^{*\circ} = b_1^i\zeta + b_2^i\zeta^2 + b_3^i\zeta^3$$

$$\left(\zeta = \frac{1}{\Delta}\int_0^y n^\circ dy, \quad \Delta = \int_0^{\Delta y} n^\circ dy, \quad t_i^{*\circ} = \frac{t_i^*}{t_{is}^*}\right) . \tag{14}$$

Here $\Delta_y = \Delta_y(x)$ is the thickness of the thermal boundary layer. The condition on the inner boundary of the layer, required for finding the coefficients of the polynomial t_i^{*0}, may be derived from (3), which gives, with $y = 0$,

$$\left[\frac{\partial}{\partial y}\left(\lambda_i\frac{\partial T_i}{\partial y}\right)\right]_w + \eta_{iw}\left(\frac{\partial u}{\partial y}\right)_w^2 + 3k\,\frac{m_e}{m}\,\frac{n_w}{\tau_{ew}}\,(T_{ew} - T_{iw}) = 0 .$$

Denoting the ion Prandtl number by

$$P_i = \frac{\eta_i c_p}{\lambda_i} = \mathrm{const}$$

and using (8), we get

$$b_2^i = -{}^1/_2\,\chi_i$$

Written in the dimensionless form, the parameter χ_i determining the shape of the ion braking temperature gradient profile is

$$\chi_i = P_i\frac{T_w}{T_s}\frac{\Delta^{\circ 2}(\varkappa-1)M^2}{K_i t_{is}^{*\circ}}\left[\left(\frac{\eta_{iw}}{\eta_{is}} - \frac{T_w}{T_s}\frac{K_i}{P_i}\right)a_1^2\frac{U_s^{\circ 2}}{\delta^{\circ 2}}\left(\frac{T_s}{T_w}\right)^2 + \frac{3}{\varkappa}\frac{m_e}{m}\frac{R_i}{M^2}\frac{n_s^\circ}{\eta_{is}^\circ}\frac{L}{U_{s0}\tau_{ew}}(T_{ew}^\circ - T_{iw}^0)\frac{T_s}{T_w}\right]$$

$$(\delta^\circ = \delta/L,\ \Delta^\circ = \Delta/L,\ \eta_{is}^\circ = \eta_{is}/\eta_{is0},\ n_s^\circ = n_s/n_{s0},\ U_s^\circ = U_s/U_{s0}$$

$$R_i = \frac{U_{s0} m n_{s0} L}{\eta_{is0}},$$

$$t_{is}^{*\circ} = t_{is}^*/T_{s0},\ T_{ew}^\circ = T_{ew}/T_{s0},\ T_{iw}^\circ = T_{iw}/T_{s0},\ M = U_{s0}\left(\frac{\varkappa k}{m}T_{s0}\right)^{-1/2} .$$

Assume that, on the outer boundary of the thermal layer ($\zeta = 1$),

$$t_i^{*\circ} = 1, \qquad \frac{\partial t_i^{*\circ}}{\partial \zeta} = 0 .$$

The following values are then obtained for the coefficients:

$$b_1^i = {}^1/_4(6 + \chi_i), \qquad b_3^i = {}^1/_4(\chi_i - 2) . \tag{15}$$

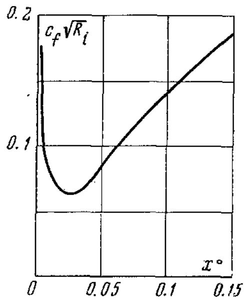

Fig. 2

The electron temperature gradient profile will be written as

$$t_e^\circ = b_1^e\zeta + b_2^e\zeta^2 + b_3^e\zeta^3 \qquad \left(t_e^\circ = \frac{t_e}{t_{es}} = \frac{T_e - T_{ew}}{T_{es} - T_{ew}}\right). \tag{16}$$

The condition on the wall is obtained from (4)

$$\left[\frac{\partial}{\partial y}\left(\lambda_e \frac{\partial T_e}{\partial y}\right)\right]_w + \left[\frac{\partial}{\partial y}\left(\frac{5}{2}\frac{j_y^q}{e} kT_e\right)\right]_w + \eta_{ew}\left(\frac{\partial u}{\partial y}\right)_w^2 + j_{xw}E_{xw}$$
$$+ j_{yw}E_{yw} - 3k\frac{m_e}{m}\frac{n_w}{\tau_{ew}}(T_{ew} - T_{iw}) = 0.$$

To transform this condition, use is made of the expression

$$j_x = \sigma\left(1 + \frac{A_2^2}{A_1^2}\right)E_x - \frac{A_2}{A_1}j_y + \sigma\frac{k}{e}\left(\frac{A_2A_3}{A_1^2} - \frac{A_4}{A_1}\right)\frac{\partial T_e}{\partial y}$$

for j_x obtained from (5) and (6), while it is assumed that

$$P_e = \frac{\eta_e c_p}{\lambda_e} = \text{const}, \qquad \frac{\eta_e\rho}{\eta_{es}\rho_s} = K_e.$$

It is assumed that the conditions

$$t_e^\circ = 1, \qquad \frac{\partial t_e^\circ}{\partial\zeta} = 0.$$

must be satisfied on the outer boundary of the thermal layer.

We obtain

$$b_1^e = \frac{3 + 0.5\chi_e}{2 - a}, \qquad b_2^e = \frac{-3a - \chi_e}{2 - a}, \qquad b_3^e = \frac{2a + 0.5\chi_e - 1}{2 - a}. \tag{17}$$

In the dimensionless form, the parameters χ_e and a, determining the shape of the electron temperature gradient profile, are

$$\chi_e = P_e\frac{T_w}{T_s}\frac{\Delta^{\circ 2}R_e}{\eta_{es}^\circ K_e t_{es}^\circ}\left\{(\varkappa - 1)M^2\frac{1}{R_e}\frac{\eta_{ew}}{\eta_{es0}}\left(\frac{T_s}{T_w}\right)^2\frac{U_s^{\circ 2}}{\delta^{\circ 2}}a_1^2\right.$$
$$+ \frac{\varkappa - 1}{\varkappa}\frac{M^j}{M}\left[\sigma_w^\circ\left(1 + \frac{A_{2w}^2}{A_{1w}^2}\right)\Phi - 2\frac{A_{2w}}{A_{1w}}\Phi + \frac{1}{\sigma_w^\circ}\Phi\right.$$
$$\left.\left. + \frac{T_{ew}}{T_w}b_1^i\frac{t_{is}^{*\circ}}{\Delta^\circ}\frac{T_s}{T_w}\right] - 3\frac{\varkappa - 1}{\varkappa}\frac{m_e}{m}\frac{T_s}{T_w}n_s^\circ\frac{L}{\tau_{ew}U_{s0}}(T_{ew}^\circ - T_{iw}^\circ)\right\}.$$
$$a = \frac{\varkappa - 1}{\varkappa}\frac{M^j}{M}\frac{R_eP_e\Delta^\circ}{2K_e}\left[\frac{5}{2}\frac{j_y^q}{j_y} + \frac{T_{ew}}{T_w} - 1 - \frac{A_{3w}}{T_w} + \sigma_w^\circ\left(\frac{A_{2w}A_{3w}}{A_{1w}^2} - \frac{A_{4w}}{A_{1w}}\right)\right].$$

The parameters K_i and K_e are approximately proportional to $(T_i/T_{is})^{3/2}$, $(T_e/T_{es})^{3/2}$, so that the present approximate method can have reasonable accuracy when T_e and T_i only vary slightly across the boundary layer.

When reducing the parameters χ_e and a to the dimensionless form, the transverse current j_y is assumed to be uniformly distributed, while no axial current is allowed outside the boundary layer; hence the scales of the current j_{ys} and the electric field-strength E_{xs} are connected by the relationship

$$j_{ys} = \sigma_{00}E_{xs}, \qquad \sigma_{00} = \sigma_s\frac{1 + A_{2s}^2/A_{1s}^2}{A_{2s}/A_{1s}}.$$

Here

$$M^j = \frac{j_y}{en_{s0}\sqrt{\varkappa T_{s0}k/m}}, \qquad \Phi = \frac{eE_{xs}L}{kT_{s0}}, \qquad R_e = \frac{U_{s0}Lmn_{s0}}{\eta_{es0}}$$

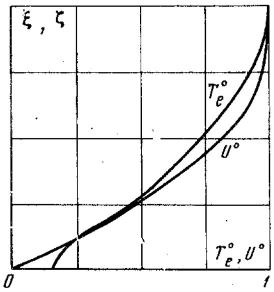

Fig. 2

$$\sigma_w{}^\circ = \frac{\sigma_w}{\sigma_{00}}, \qquad t_{es}{}^\circ = \frac{t_{es}}{T_{s0}}, \qquad \eta_{es}{}^\circ = \frac{\eta_{es}}{\eta_{es0}}$$

and account is taken of the fact that

$$j_y = j_{ys}, \; E_x = E_{xs}.$$

The quantity $j_y{}^q/j_y$ appearing in χ_e is given by

$$j_y{}^{q\circ} = \frac{j_y{}^q}{j_y} = \frac{A_{7w}}{A_{1w}}\left(1 - \frac{A_{9w}}{A_{1w}}\sigma_w{}^\circ - \frac{A_{3w}}{A_{1w}}\sigma_w{}^\circ \frac{t_{es}{}^\circ}{\Delta^\circ} b_1{}^e \frac{T_s}{T_w}\frac{1}{\Phi}\right) + \frac{A_{8w}}{A_{1w}}\sigma_w{}^\circ.$$

Having obtained the ion and electron temperature gradient profiles, we can write the profile of the plasma braking temperature gradients in the boundary layer in the form

$$t^{*\circ} = b_1\zeta + b_2\zeta^2 + b_3\zeta^3 \tag{18}$$

where

$$\begin{gathered} t^{*\circ} = \frac{t^*}{t_s{}^*}, \quad t^* = T^* - T_w, \quad t_s{}^* = T_s{}^* - T_w, \\ T^* = T + \frac{u^2}{2c_p}, \quad T_s{}^* = T_s + \frac{u_\Delta{}^2}{2c_p} \\ b_1 = b_1{}^e\frac{t_{es}}{t_s{}^*} + b_1{}^i\frac{t_{is}{}^*}{t_s{}^*}, \quad b_2 = b_2{}^e\frac{t_{es}}{t_s{}^*} + b_2{}^i\frac{t_{is}{}^*}{t_s{}^*}, \quad b_3 = b_3{}^e\frac{t_{es}}{t_s{}^*} + b_3{}^i\frac{t_{is}{}^*}{t_s{}^*}. \end{gathered} \tag{19}$$

On integrating the equation of motion across the layer and recalling (10), we can reduce it to the ordinary equation

$$(\rho_s U_s{}^2)'\vartheta + \rho_s U_s{}^2\vartheta' + \rho_s U_s U_s'\left[\delta^* - \Delta^*\left(1 + \frac{u_\Delta{}^2}{2c_pT_s} - \frac{T_w}{T_s}\right) + \frac{u_\Delta{}^2}{2c_pT_s}(\delta^* + \vartheta)\right] = \tau_{iw}. \tag{20}$$

Here

$$\begin{gathered} \vartheta = \int_0^\infty \rho^\circ u^\circ(1 - u^\circ)\,dy = \delta\int_0^\infty u^\circ(1 - u^\circ)\,d\xi \\ \delta^* = \int_0^\infty \rho^\circ(1 - u^\circ)\,dy = \delta\int_0^\infty (1 - u^\circ)\,d\xi \\ \Delta^* = \int_0^\infty \rho^\circ(1 - t^{*\circ})\,dy = \Delta\int_0^\infty (1 - t^{*\circ})\,d\zeta \\ \tau_{iw} = \eta_{iw}\left(\frac{\partial u}{\partial y}\right)_w = \eta_{iw}a_1\frac{U_s}{\delta}\frac{T_s}{T_w}. \end{gathered}$$

Adding Eqs. (3) and (4), we find the energy equation for the plasma as a whole to be

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\lambda_i\frac{\partial T_i}{\partial y}\right) + \frac{\partial}{\partial y}\left(\lambda_e\frac{\partial T_e}{\partial y} + \frac{5}{2}\frac{j_y{}^q}{e}kT_e\right) + u\frac{\partial p}{\partial x} + \eta_i\left(\frac{\partial u}{\partial y}\right)^2 + j_xE_x + j_y(E_y - uB). \tag{21}$$

We add this equation to Eq. (2), as multiplied by u, to get

$$\rho c_p\left(u\frac{\partial T^*}{\partial x} + v\frac{\partial T^*}{\partial y}\right) = \frac{\partial}{\partial y}\left(\lambda_i\frac{\partial T_i}{\partial y}\right) + \frac{\partial}{\partial y}\left(\lambda_e\frac{\partial T_e}{\partial y} + \frac{5}{2}\frac{j_y{}^q}{e}kT_e\right) + \frac{\partial}{\partial y}(\tau_i u) + j_xE_x + j_yE_y. \tag{22}$$

Integrating (22) across the layer under the assumption that T_W = const, we can reduce it to the form

$$(\rho_s^2 U_s^2 t_s^{*2} \theta^2)' = 2\rho_s U_s t_s^* \Big\{ \rho_s U_s t_s^{*\prime} (\delta - \delta^*)\theta + \frac{\lambda_{iw}}{c_p} b_1{}^i t_{is}^* \frac{T_s}{T_w} \frac{\theta}{\Delta} + \frac{\lambda_{ew}}{c_p} b_1{}^e t_{es} \frac{T_s}{T_w}\frac{\theta}{\Delta} - \frac{5}{2}\frac{k}{ec_p}(j_{ys}{}^q T_{es} - j_{yw}{}^q T_{ew})\theta - \frac{1}{c_p}\theta \int_0^{\Delta \bar{y}} (j_x E_x + j_y E_y)\,dy \Big\} \tag{23}$$

where

$$\theta = \int_0^\infty \rho^\circ u^\circ (1 - t^{*\circ})\,dy = \Delta \int_0^\infty u^\circ (1 - t^{*\circ})\,d\zeta$$

is the thickness of the energy loss.

The mean Joule dissipation over the layer will be found from (5) and (6) in conjunction with Eq. (18) for the temperature profile, under the simplifying assumptions that

$$\sigma \approx \sigma_w, \quad H_e \approx H_{ew}, \quad T_e \approx T_{ew}, \quad u/U_s = \xi\,.$$

After reduction to the dimensionless form, (23) then becomes

$$(\rho_s^{\circ 2}\, U_s^{\circ 2}\, t_s^{*\circ 2}\, \theta^{\circ 2})' = 2\Phi_\Delta\, \rho_s^\circ U_s^\circ\, t_s^{*\circ} \tag{24}$$

where

$$\begin{aligned}
\Phi_\Delta = {} & t_s^{*\circ\prime} \rho_s^\circ U_s^\circ \theta^\circ (\delta^\circ - \delta^{*\circ}) + \Big(\frac{1}{R_i P_i} \frac{\lambda_{is}}{\lambda_{is0}} \frac{\lambda_{iw}}{\lambda_{is}} b_1{}^i t_{is}^{*\circ} \\
& + \frac{1}{R_e P_e} \frac{\lambda_{es}}{\lambda_{es0}} \frac{\lambda_{ew}}{\lambda_{es}} b_1{}^e t_{es}^\circ \Big) \frac{T_s}{T_w} \frac{\theta}{\Delta} - (j_{ys}{}^{q\circ} T_{es}^\circ - j_{yw}{}^{q\circ} T_{ew}^\circ) \frac{M^j}{M} \theta^\circ \\
& - \theta^\circ \Delta^\circ \Big[\sigma_w^\circ \Big(1 + \frac{A_{2w}^2}{A_{1w}^2} \Big) - 2\frac{A_{2w}}{A_{1w}} + \frac{1}{\sigma_w^\circ} \Big] \Big\{ 1 + \frac{\varkappa - 1}{2} M^2 \frac{U_s^{\circ 2}}{T_s^\circ} \Big[u_\Delta^{\circ 2} \\
& - \frac{1}{3}\Big(\frac{\Delta}{\delta}\Big)^2 \Big] - \Big(1 + \frac{\varkappa - 1}{2} M^2 u_\Delta^{\circ 2} \frac{U_s^{\circ 2}}{T_w^\circ} - \frac{T_w}{T_s} \Big) \frac{\Delta^*}{\Delta} \Big\} \frac{\varkappa - 1}{\varkappa} \frac{M^j}{M} \Phi \\
& + \theta^\circ (T_{es}^\circ - T_{ew}^\circ) \frac{\varkappa - 1}{\varkappa} \frac{M^j}{M} \Big[\sigma_w^\circ \Big(\frac{A_{4w}}{A_{1w}} - \frac{A_{2w} A_{3w}}{A_{1w}^2} \Big) + \frac{A_{3w}}{A_{1w}} + 1 \Big] \\
& - \theta^\circ \Delta^\circ \frac{U_s^\circ}{K} \frac{\varkappa - 1}{\varkappa} \frac{M^j}{M} \Phi \Big\{ \frac{1}{2}\frac{\Delta}{\delta} + \frac{\varkappa - 1}{2} M^2 \frac{U_s^{\circ 2}}{T_w^\circ} \Big[\frac{u_\Delta^{\circ 2}}{2} \frac{\Delta}{\delta} - \frac{1}{4}\Big(\frac{\Delta}{\delta}\Big)^3 \Big] \\
& - \Big(1 + \frac{\varkappa - 1}{2} M^2 u_\Delta^{\circ 2} \frac{U_s^{\circ 2}}{T_w^\circ} - \frac{T_w}{T_s} \Big) \frac{\theta}{\Delta} \Big\} - \theta \frac{\varkappa - 1}{\varkappa} \frac{M^j}{M} \Big(\ln \frac{T_s}{T_w} \Big) T_{ew}^\circ\,.
\end{aligned}$$

Here

$$\begin{gathered}
t_s^{*\circ} = \frac{T_s}{T_{s0}} + \frac{\varkappa - 1}{2} M^2 U_s^{\circ 2} \Big(\frac{u_\Delta}{U_s}\Big)^2 - \frac{T_w}{T_{s0}}, \quad u_\Delta^\circ = \frac{u_\Delta}{U_s} \qquad t_{is}^{*\circ} = \frac{T_{is}}{T_{s0}} \\
+ \frac{\varkappa - 1}{2} M^2 U_s^{\circ 2} \Big(\frac{u_\Delta}{U_s}\Big)^2 - \frac{T_{iw}}{T_{s0}}, \quad K = \frac{E_{xs}}{BU_{s0}} \\
u_\Delta^\circ = a_1 \frac{\Delta}{\delta} + a_2 \Big(\frac{\Delta}{\delta}\Big)^2 + a_3 \Big(\frac{\Delta}{\delta}\Big)^3 + a_4 \Big(\frac{\Delta}{\delta}\Big)^4, \quad ' = \frac{d}{dx^\circ}, \quad x^\circ = \frac{x}{L}\,.
\end{gathered}$$

Using profiles (8) and (18) to evaluate the integral thicknesses of the boundary layer, it is found that

$$\begin{gathered}
\delta^* = \vartheta H = \delta \frac{8 - a_1}{20}, \qquad \vartheta = \delta \frac{-5a_1^2 + 12a_1 + 144}{1260} \\
\Delta^* = \Delta (1 - {}^1/_2 b_1 - {}^1/_3 b_2 - {}^1/_4 b_3) \\
\theta = \Delta \Big\{ a_1 \frac{\Delta}{\delta} \Big(\frac{1}{2} - \frac{1}{3} b_1 - \frac{1}{4} b_2 - \frac{1}{5} b_3 \Big)
\end{gathered} \tag{25}$$

$$+ a_2 \Big(\frac{\Delta}{\delta}\Big)^2 \Big(\frac{1}{3} - \frac{1}{4} b_1 - {}^1/_5 b_2 - {}^1/_6 b_3 \Big) + a_3 (\Delta/\delta)^3 ({}^1/_4 - {}^1/_5 b_1 - {}^1/_6 b_2 - {}^1/_7 b_3) + a_4 (\Delta/\delta)^4 ({}^1/_5 - {}^1/_6 b_1 - {}^1/_7 b_2 - {}^1/_8 b_3) \Big\}\,.$$

Following [2], the integral relation (20) for the moments may be reduced to a quadrature which is convenient for practical computations. We write (20) in the form

$$2\vartheta^\circ \vartheta^{\circ\prime} + c_1 \frac{U_s^{\circ\prime}}{U_s^\circ} \vartheta^{\circ 2} + 2 \frac{U_s^{\circ\prime}}{U_s^\circ} \Big(\frac{T_w}{T_s} - 1 \Big) \frac{\Delta^*}{\vartheta} \vartheta^{\circ 2} \tag{26}$$

$$
+2\frac{U_s^{\circ\prime}}{U_s}\frac{\varkappa-1}{2}M^2u_\Delta^{\circ 2}\frac{U^{\circ 2}}{T_s^\circ}\left(H+1-\frac{\Delta^*}{\vartheta}\right)\vartheta^{\circ 2}+2\frac{\rho_s^{\circ\prime}}{\rho_s^\circ}\vartheta^{\circ 2}
= 2a_1\frac{\eta_{iw}}{\eta_{is0}}\frac{1}{\rho_s^\circ U_s^\circ R_i}\frac{T_s}{T_w}\frac{\vartheta}{\delta}-\frac{U_s^{\circ\prime}}{U_s^\circ}\vartheta^{\circ 2}\left[2(H+2)-c_1\right] \qquad (26)
$$

where c_1 is a constant, to be defined below. Put

$$
c=c_1+2\left(\frac{T_w}{T_s}-1\right)\frac{\Delta^*}{\vartheta}\approx \text{const}, \qquad k=(\varkappa-1)\,M^2\frac{1}{T_s^\circ}\left(H+1-\frac{\Delta^*}{\vartheta}\right)u_\Delta^{\circ 2}\approx \text{const}
$$

and denote

$$
\Phi=2a_1\frac{\vartheta}{\delta}-\lambda\left(\frac{\vartheta}{\delta}\right)^2K_i\frac{\eta_{is}}{\eta_{iw}}\left[2(H+2)-c_1\right]. \qquad (27)
$$

Multiplying (26) by $U^{\circ c}$, we can write it in the linear form

$$
(\vartheta^{\circ 2}U_s^{\circ c})'+\left(kU_s^\circ U_s^{\circ\prime}+2\frac{\rho_s^{\circ\prime}}{\rho_s^\circ}\right)\vartheta^{\circ 2}U_s^{\circ c}=U_s^{\circ\,(c-1)}\frac{T_s}{T_w}\frac{1}{R_i}\Phi\frac{\eta_{iw}^\circ}{\rho_s^\circ}.
$$

The solution is

$$
\vartheta^{\circ 2}U_s^{\circ c}R_i=e^{\frac{-kU_s^{\circ 2}}{2}}\rho_s^{\circ\,(-2)}\left(\text{const}+\int_{x_0^\circ}^{x^\circ}U_s^{\circ\,(c-1)}\rho_w^\circ\eta_{iw}^\circ e^{\frac{kU_s^{\circ 2}}{2}}\Phi dx^\circ\right). \qquad (28)
$$

Successive approximations have to be used to obtain ϑ_0 from (28). Rapid convergence is ensured if c_1 is chosen in such a way that the function Φ varies weakly with λ.

According to (27), Φ is a nearly linear function of λ; on demanding that the coefficient of λ vanish, the following relationship is obtained for c_1:

$$
c_1=2(H+2)_{\lambda=0}-\frac{1}{3(\vartheta/\delta)_{\lambda=0}}\frac{\eta_{iw}}{\eta_{is}}\frac{1}{K_i}.
$$

When $\Phi\approx$ const, the basic equation (28) amounts to a simple quadrature.

Since Φ_Δ is not constant, the solution of the integral relation (24) is best written in the form

$$
(\rho_s^{\circ 2}U_s^{\circ 2}t_s^{*\circ 2}\theta^{\circ 2})_{x_k^\circ}=2(\Phi_\Delta)_{x^\circ=0}\int_0^{x_1^\circ}\rho_s^\circ U_s^\circ t_s^{*\circ}dx^\circ
+2(\Phi_\Delta)_{x_1^\circ}\int_{x_1^\circ}^{x_2^\circ}\rho_s^\circ U_s^\circ t_s^{*\circ}dx^\circ+\ldots+2(\Phi_\Delta)_{x_k^\circ}\int_{x_k^\circ}^{x_{k+1}^\circ}\rho_s^\circ U_s^\circ t_s^{*\circ}dx^\circ+\ldots. \qquad (29)
$$

It is assumed that Φ_Δ is fixed in each subinterval of the boundary layer, from x°_k to x°_{k+1}; its value is found on the assumption that the relevant quantities take the values that they have at the start of the subinterval.

The computation starts with finding the quantities at the channel input, i.e., at the start of the first subinterval. If $U^\circ_s \neq 0$ here (there is no critical point), and if no auxiliary information is available on the initial thicknesses, it may be assumed that

$$
\delta=\Delta=\delta^*=\vartheta=\Delta^*=\theta=0 \qquad \text{when } x^\circ=x_0=0\,.
$$

To obtain the initial condition with $x^\circ = 0$, l'Hôpital's rule has to be applied to the indeterminate forms and the initial values of Δ/δ, θ/Δ and the other ratios determined.

With x = 0, (20) gives

$$\frac{d}{dx^\circ}(\rho_s^\circ U_s^{\circ 2}\vartheta^\circ)^2 = 2\rho_s^\circ U_s^{\circ 3}\frac{T_s}{T_w}a_1\frac{\vartheta}{\delta}\frac{\eta_{iw}}{\eta_{is0}}\frac{1}{R_i},$$

From (24), with x = 0,

$$\frac{d}{dx^\circ}(\rho_s^\circ U_s^\circ t_s^{*\circ}\theta^\circ)^2 = 2\rho_s^\circ U_s^\circ t_s^{\circ *}\left[\frac{1}{R_iP_i}\frac{\lambda_{iw}}{\lambda_{is0}}b_1^i t_{is}^{*\circ} + \frac{1}{R_eP_e}\frac{\lambda_{ew}}{\lambda_{es0}}b_1^e t_{es}^\circ\right]\frac{T_s}{T_w}\frac{\theta}{\Delta}.$$

Applying l'Hôpital's rule,

$$\left(\frac{\theta}{\Delta}\right)\left(\frac{\Delta}{\delta}\right)^2 = \left(\frac{\vartheta}{\delta}\right)\left[\frac{1}{R_iP_i}\frac{\lambda_{iw}}{\lambda_{is0}}b_1^i t_{is}^{*\circ} + \frac{1}{R_eP_e}\frac{\lambda_{ew}}{\lambda_{es0}}b_1^e t_{es}^\circ\right]\left(2\frac{\eta_{iw}}{\eta_{is0}}\frac{1}{R_i}t_s^{*\circ}\right)^{-1}.$$

This equation, in conjunction with (25), (19), (15), (17), and (27), provides the basis for the numerical computation of the initial ratio Δ/δ.

The friction and the heat exchange are found from the computation results for the distributions $\delta = \delta(x)$ and $\Delta = \Delta(x)$. The coefficient of ion friction is

$$c_{f_i}\sqrt{R_i} = \frac{2\tau_{iw}}{mn_{s0}U_{s0}^2} = 2a_1\frac{\eta_{iw}}{\eta_{is0}}\frac{T_s}{T_w}\frac{U_s^\circ}{\delta^\circ}.$$

Stanton's number for the ions is

$$S_i = \frac{-q_w^i}{mn_{s0}U_{s0}c_pt_{s0}^*} = b_1^i\frac{\lambda_{iw}}{\lambda_{is0}}\frac{T_s}{T_w}\frac{t_{is}^*}{t_{s0}^*}\frac{1}{\Delta^\circ R_iP_i}.$$

Stanton's number for the electrons is determined by the thermal conductivity and the convection:

$$S_e = \frac{-q_w^e}{mn_{s0}U_{s0}c_pt_{s0}^*} = b_1^e\frac{\lambda_{ew}}{\lambda_{es0}}\frac{T_s}{T_w}\frac{t_{es}}{t_{s0}^*}\frac{1}{\Delta^\circ R_eP_e}.$$

Stanton's number for the transport of enthalpy by the electric current is

$$S_j = \frac{5}{2}\frac{kT_{ew}j_{yw}^q}{emn_{s0}U_{s0}c_pt_{s0}^*} = \frac{5}{2}\frac{\varkappa-1}{\varkappa}\frac{T_{ew}}{t_{s0}^*}\frac{j_y^q}{j_y}\frac{M^j}{M}.$$

As an example, the boundary layer of a two-temperature argon plasma at the positive electrode of an accelerator channel was calculated under the conditions:

characteristic dimension (channel width) L = 0.2 m, entry velocity U_{s0} = 5000 m/sec, pressure p = 10^{-2} mm Hg, ion temperature T_{is_0} = 2000° K, electron temperature T_{es0} = 10,000°K, T_{ew} = 1500° K, T_{iw} = 300° K, and magnetic field induction B = 0.5 T. These figures correspond to

$$n_{s0} = 7.794\cdot 10^{18}\text{m}^{-3},\quad \tau_{es0} = 4.99\cdot 10^{-9}\text{ sec},\quad \tau_{is0} = 2.2\cdot 10^{-7}\text{ sec}.$$
$$n_w = 5.196\cdot 10^{19}\text{m}^{-3},\quad \tau_{ew} = 9.34\cdot 10^{-11}\text{ sec},\quad \tau_{iw} = 6.23\cdot 10^{-9}\text{ sec}.$$

The figure

$$S_m = \frac{j_{ys}BL}{mn_{s0}U_{s0}^2} = 50$$

was given, corresponding to a discharge current density j_y = 6459 · 3A/m² and electric field-strength E_x = j_y/σ_{00} = 2590 v/m.

The corresponding figures for the criteria are

$$K = \frac{E_{xs}}{BU_{s0}} = 1.036, \quad M^j = \frac{j_y}{en_{s0}\sqrt{\varkappa (k/m) T_{s0}}} = 2.538$$

$$\Phi = \frac{eE_{xs}L}{kT_{s0}} = 500, \quad H_{es0} = 438, \quad H_{is0} = 0{,}265$$

$$R_i = 1.565 \cdot 10^4, \; R_e = 3.6 \cdot 10^{10}, \; P_i = 0.619, \; P_e = 1.925 \cdot 10^{-5}.$$

The case of an elementary external flow was selected, i.e., an isothermal plasma flow with breakaway of the electron temperature

$$T_s = T_{s0} = \text{const}, \quad T_{is} = T_{is0} = \text{const}, \quad T_{es} = T_{es0} = \text{const}$$

with constant pressure, with uniformly distributed discharge current, and with forbidden axial current

$$j_{ys} = j_{ys0} = \text{const}, \quad j_{xs} = 0.$$

The expression $U_s{}^\circ = \sqrt{1 + 2S_m x^\circ}$ was then obtained from the equation of motion (2).

The distributions of the dynamic thicknesses δ, δ^*, and ϑ over the length of the electrode, and the distributions of Δ, Δ^*, and θ, are shown in Fig. 1.

A curve of $C_f\sqrt{R_i}$ is given in Fig. 2. Profiles of the velocity $u^\circ = u/U_s$ and the temperature $T_e{}^\circ = T_e/T_{es}$, are given in Fig. 3.

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